

Stochastic Verification Theorem of Forward-Backward Controlled Systems for Viscosity Solutions

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Abstract

In this paper, we investigate the controlled systems described by forward-backward stochastic differential equations with the control contained in drift, diffusion and generator of BSDEs. A new verification theorem is derived within the framework of viscosity solutions without involving any derivatives of the value functions. It is worth pointing out that this theorem has wider applicability than the restrictive classical verification theorems. As a relevant problem, the optimal stochastic feedback controls for forward-backward systems are discussed as well.

Key words: Stochastic optimal control, forward-backward stochastic differential equations, H-J-B equations, viscosity solutions, super/sub-differentials, optimal feedback controls.

1 Introduction

Since the fundamental work of Pardoux & Peng [1], the theory of BSDEs and FBSDEs have become a powerful tool in many fields, such as mathematics finance, optimal control, stochastic games, partial differential equations and homogenization etc. Recently, the partially coupled FBSDEs controlled systems have been studied in [2], [3], and [4], where the authors used the dynamic programming principle and proved that the value function is to be the unique viscosity solution of the H-J-B equations. In [5], the authors investigated the

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existence of an optimal control for forward-backward control systems using a verification theorem, of course, under smooth situation. Hence, as an important part in viscosity theory, a natural question arises: do verification theorems still hold, with the solutions of H-J-B equations in the classical sense replaced by the ones in the viscosity sense and the derivatives involved replaced by the super-differentials or sub-differentials? For the deterministic and forward stochastic cases, the answer to the above questions is "positive". For more details, see [6], [7], [8] and [9].

The present paper proceeds to give the answer to the above question for forward-backward stochastic systems.

Throughout this paper, we denoted by \mathbf{R}^n the space of n -dimensional Euclidean space, by $\mathbf{R}^{n \times d}$ the space the matrices with order $n \times d$, by \mathbf{S}^n the space of symmetric matrices with order $n \times n$. $\langle \cdot, \cdot \rangle$ and $|\cdot|$ denote the scalar product and norm in the Euclidean space, respectively. $*$ appearing in the superscripts denoted the transpose of a matrix.

Let $T > 0$ and let (Ω, \mathcal{F}, P) be a complete probability space, equipped with a d -dimensional standard Brownian motion $\{W(t)\}_{0 \leq t \leq T}$. For a given $s \in [t, T]$, we suppose that the filtration $\{\mathcal{F}_t^s\}_{s \leq t \leq T}$ is generated as the following

$$\mathcal{F}_t^s = \sigma \{W(r) - W(s); s \leq r \leq T\} \vee \mathcal{N},$$

where \mathcal{N} contains all P -null sets in \mathcal{F} . In particular, if $s = 0$ we write $\mathcal{F}_t = \mathcal{F}_t^s$.

Let \mathcal{X} be a Hilbert space with the norm $\|\cdot\|_{\mathcal{X}}$, and p , $1 \leq p \leq +\infty$, define the set $L_{\mathcal{F}}^p(a, b; \mathcal{X}) = \{\phi(\cdot) = \{\phi(t, \omega) : a \leq t \leq b\} | \phi(\cdot) \text{ is an } \mathcal{F}_t\text{-adapted, } \mathcal{X}\text{-valued measurable process on } [a, b], \text{ and } \mathbf{E} \int_a^b \|\phi(t, \omega)\|_{\mathcal{X}}^p dt < +\infty\}$.

Let U is a given closed set in some Euclidean space \mathbf{R}^m . For a given $s \in [0, T]$, we denote by $\mathcal{U}_{ad}(s, T)$ the set of U -valued \mathcal{F}_t^s -predictable processes. For any initial time $s \in [t, T]$ and initial state $y \in \mathbf{R}^d$, we consider the following stochastic control systems

$$\begin{cases} dX^{s,y;u}(t) = b(t, X^{s,y;u}(t), u(t)) dt + \sigma(t, X^{s,y;u}(t), u(t)) dW_t, \\ dY^{s,y;u}(t) = -f(t, X^{s,y;u}(t), Y^{s,y;u}(t), Z^{s,y;u}(t), u(t)) dt + Z^{s,y;u}(t) dW_t, \\ X^{s,y;u}(s) = x, \quad Y^{s,y;u}(T) = \Phi(X^{s,y;u}(T)). \end{cases} \quad (1.1)$$

where

$$\begin{aligned} b &: \mathbf{R}^d \times U \rightarrow \mathbf{R}^d, \\ \sigma &: \mathbf{R}^d \times U \rightarrow \mathbf{R}^{d \times d}, \\ f &: [0, T] \times \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^d \times U \rightarrow \mathbf{R}, \\ \Phi &: \mathbf{R}^d \rightarrow \mathbf{R}. \end{aligned}$$

They satisfy the following conditions

(H1) b and σ are continuous in t .

(H2) For some $L > 0$, and all $x, x' \in \mathbf{R}^d$, $v, v' \in U$, a.s.

$$\left| b(t, x, v) - b(t, x', v') \right| + \left| \sigma(t, x, v) - \sigma(t, x', v') \right| \leq L \left(|x - x'| + |v - v'| \right).$$

Obviously, under the above assumptions, for any $v(\cdot) \in \mathcal{U}_{ad}$, the first control system of (1.1) has a unique strong solution

$$\{X^{s,y;u}(t), 0 \leq s \leq t \leq T\}.$$

(H3) f and Φ are continuous in t .

(H4) For some $L > 0$, and all $x, x' \in \mathbf{R}^d, y, y' \in \mathbf{R}, z, z' \in \mathbf{R}^d, v, v' \in U$, a.s.

$$\begin{aligned} & \left| f(t, x, y, z, v) - f(t, x', y', z', v') \right| + \left| \Phi(x) - \Phi(x') \right| \\ & \leq L \left(\left| x - x' \right| + \left| y - y' \right| + \left| z - z' \right| + \left| v - v' \right| \right). \end{aligned}$$

From the classical theory of BSDEs, we claim that there exists a triple $(X^{s,y;u}, Y^{s,y;u}, Z^{ts,y;u})$, which is the unique solution of the FBSDEs (1.1).

Given a control process $u(\cdot) \in \mathcal{U}_{ad}(s, T)$ we consider the following cost functional

$$J(s, y; u(\cdot)) = Y^{s,y;u}(s), \quad (s, y) \in [0, T] \times \mathbf{R}^d, \quad (1.2)$$

where the process $Y^{s,y;u}$ is defined by FBSDEs (1.1). It follows from the uniqueness of the solution of the SDEs and BSDEs that

$$\begin{aligned} & Y^{s,y;u}(s + \delta) \\ & = Y^{s+\delta, X^{s,y;u}(t+\delta);u}(s + \delta) \\ & = J(t + \delta, X^{s,y;u}(t + \delta)), \quad \text{a.s.} \end{aligned}$$

The object of the optimal control problem is to minimize the cost function $J(s, y; u(\cdot))$, for a given $(s, y) \in [0, T] \times \mathbf{R}^d$, over all $u(\cdot) \in \mathcal{U}_{ad}(s, T)$. We denote the above problem by $C_{s,y}$ to recall the dependence on the initial time s and the initial state y . The value function is defined as

$$V(s, y) = \inf_{u(\cdot) \in \mathcal{U}_{ad}(s, T)} J(s, y; u(\cdot)). \quad (1.3)$$

An admissible pair $(X^*(\cdot), u^*(\cdot))$ is called optimal for $C_{s,y}$ if $u^*(\cdot)$ achieves the minimum of $J(s, y; u(\cdot))$ over $\mathcal{U}_{ad}(s, T)$.

As we have known that the verification technique plays an important role in testing for optimality of a given admissible pair and, especially, in constructing optimal feedback controls. Let us recall the similar classical verification theorem as follows.

Theorem 1. *Let $W \in C^{1,2}([0, T] \times \mathbf{R}^d)$ be a solution of the following Hamilton-Jacobi-Bellman (H-J-B) equations:*

$$\begin{cases} \frac{\partial}{\partial t} W(t, x) + H_0(t, x, W, DW, D^2W) = 0, & (t, x) \in [0, T] \times \mathbf{R}^d, \\ W(T, x) = \Phi(x), & x \in \mathbf{R}^d. \end{cases} \quad (1.4)$$

The Hamiltonian is given by

$$H_0(t, x, W, DW, D^2W) = \inf_{u \in U} H(t, x, W, DW, D^2W, u),$$

where

$$\begin{aligned} & H(t, x, \Psi, D\Psi, D^2\Psi, u) \\ &= \frac{1}{2} \text{tr}(\sigma \sigma^*(t, x, u) D^2\Psi) + \langle D\Psi, b(t, x, u) \rangle \\ & \quad + f(t, x, \Psi(t, x), D\Psi(t, x) \cdot \sigma(t, x, u), u), \\ (t, x, u) &\in [0, T] \times \mathbf{R}^d \times U, \\ \Psi &\in C^{1,2}([0, T] \times \mathbf{R}^d). \end{aligned}$$

Here the function b, σ, f and Φ are supposed to satisfy (H1)-(H4). Then

1°)

$$W(s, y) \leq J(s, y; u(\cdot))$$

for any $(s, y) \in [0, T] \times \mathbf{R}^d$ and $u(\cdot) \in \mathcal{U}_{ad}(s, T)$.

2°) Supposed that a given admissible pair $(x^*(\cdot), u^*(\cdot))$, here $x^*(\cdot) = X^*(\cdot)$, for the problem $C_{s,y}$ satisfies

$$\begin{aligned} & \frac{\partial}{\partial t} W(t, x^*(t)) \\ & + H(t, x^*(t), W(t, x^*(t)), DW(t, x^*(t)), D^2W(t, x^*(t)), u^*(t)) \\ & = 0, \quad P\text{-a.s., a.e. } t \in [s, T]; \end{aligned} \tag{1.5}$$

then $(x^*(\cdot), u^*(\cdot))$ is an optimal pair for the problem $C_{s,y}$.

The proof follows from Theorem 9 in Section 3 in our paper.

Remark 2. By H-J-B equations, (1.5) is equivalent to the following form

$$\begin{aligned} & \min_{u \in U} H(t, x^*(t), W(t, x^*(t)), DW(t, x^*(t)), D^2W(t, x^*(t)), u) \\ & = H(t, x^*(t), W(t, x^*(t)), DW(t, x^*(t)), D^2W(t, x^*(t)), u^*(t)). \end{aligned}$$

Then, an optimal feedback control $u^*(t, x)$ can be constructed by minimizing

$$H(t, x, W(t, x), DW(t, x), D^2W(t, x), u)$$

over $u \in U$.

Remark 3. We claim that (1.5) is equivalent to

$$W(s, y) = J(s, y; u^*(\cdot)).$$

Actually, we have

$$\begin{aligned} & \Phi(X^*(T)) - W(s, y) \\ = & W(T, X^*(T)) - W(s, y) \\ = & \int_s^T \frac{d}{dt} W(t, x^*(t)) dt \\ = & \int_s^T \left[\frac{\partial}{\partial t} W(t, x^*(t)) \right. \\ & + H(t, x^*(t), W(t, x^*(t)), DW(t, x^*(t)), D^2W(t, x^*(t)), u^*(t)) \\ & - f(t, x^*(t), W(t, x), DW(t, x^*(t)) \cdot \sigma(t, x^*(t), u^*(t)), u^*(t))] dt \\ & \left. + \int_s^T W_x(t, x^*(t)) \cdot \sigma(t, x^*(t), u^*(t)) dW_t \right], \end{aligned}$$

which implies

$$\begin{aligned} & W(s, y) \\ = & J(s, y; u^*(\cdot)) + \int_s^T \left[\frac{\partial}{\partial t} W(t, x^*(t)) \right. \\ & \left. + H(t, x^*(t), W(t, x^*(t)), DW(t, x^*(t)), D^2W(t, x^*(t)), u^*(t)) \right] dt \end{aligned}$$

It is necessary to point out that in Theorem 1 we need $W \in C^{1,2}([0, T] \times \mathbf{R}^d)$. However, when we take the verification function W to be the value function V , as V satisfies the HJB equations if $V \in C^{1,2}([0, T] \times \mathbf{R}^d)$. Unfortunately, in general the H-J-B equations (1.4) do not admit smooth solutions, which makes the applicability of the classical verification theorem very restrictive and is a major deficiency in dynamic programming theory. As we have known that the viscosity theory of nonlinear PDEs was launched by Crandall and Lions. In this theory, all the derivatives involved are replaced by the super-differentials and sub-differentials, and solution in viscosity sense can be only continuous function (For more information see in [10]). Besides, since the verification theorems can be played primary roles in constructing optimal feedback controls, while in many practical problems H-J-B equations do not admit smooth solutions, hence, we want to answer the question aforementioned.

Our paper is organized as follows: In Section 2, we introduce some preliminary results about viscosity solutions and the associated the second order one-sided super/sub-differentials. In Section 3, a new verification theorem in term of viscosity solutions and the super-differentials are established. At last, we show the way to find the optimal feedback controls in Section 4.

2 Super-differentials, Sub-differentials, and Viscosity Solutions

Let Q be an open subset of \mathbf{R}^n , and $v : \overline{Q} \rightarrow \mathbf{R}$ be a continuous function.

Definition 4. *The second order one-sided super-differentials (resp., sub-differentials) of v at $(t_0, x_0) \in [0, T) \times \mathbf{R}^n$, denoted by $D_{t+,x}^+ v(t_0, x_0)$ (resp. $D_{t+,x}^- v(t_0, x_0)$), is a set defined by*

$$\begin{aligned} & D_{t+,x}^+ v(t_0, x_0) \\ = & \{ (p, q, \Theta) \in \mathbf{R} \times \mathbf{R}^d \times \mathbf{S}^d \mid \\ & \overline{\lim}_{t \rightarrow t_0+, x \rightarrow x_0} \frac{v(t, x) - v(t_0, x_0) - p(t - t_0) - \langle q, x - x_0 \rangle - \frac{1}{2}(x - x_0)^* Q(x - x_0)}{|t - t_0| + |x - x_0|^2} \\ & \leq 0 \}. \end{aligned}$$

by (resp.,

$$\begin{aligned} & D_{t+,x}^- v(t_0, x_0) \\ = & \{ (p, q, \Theta) \in \mathbf{R} \times \mathbf{R}^d \times \mathbf{S}^d \mid \\ & \underline{\lim}_{t \rightarrow t_0+, x \rightarrow x_0} \frac{v(t, x) - v(t_0, x_0) - p(t - t_0) - \langle q, x - x_0 \rangle - \frac{1}{2}(x - x_0)^* Q(x - x_0)}{|t - t_0| + |x - x_0|^2} \\ & \geq 0 \}. \end{aligned}$$

Let us recall the definition of a viscosity solution for (1.4) from [3] or [4]

Definition 5. *An continuous function v on $[0, T] \times \mathbf{R}^n$ is called a viscosity subsolution (resp., supersolution) of the H-J-B equations (1.4) if*

$$v(T, x) \leq \Phi(x).$$

and

$$\frac{\partial \varphi}{\partial t}(t_0, x_0) + \inf_{u \in U} \{ H(t_0, x_0, \varphi(t_0, x_0), D\varphi(t_0, x_0), D^2\varphi(t_0, x_0), u) \} \geq (\leq) 0 \quad (2.1)$$

whenever $v - \varphi$ attains a local maximum (resp., minimum) at (t_0, x_0) in a right neighborhood of (t_0, x_0) for $\varphi \in C^{1,2}([0, T] \times \mathbf{R}^n)$. A function v is called a viscosity solution of (1.4) if it is both a viscosity subsolution and a supersolution of (1.4).

The equivalence of Definition 4 and the Definition 5 in which derivatives of test functions are replaced by elements of the second order one-sided sub- and super-differentials are established with the help of a well-known result that we present below and whose proof can be found in [13].

Lemma 6. Let $(t_0, x_0) \in [0, T] \times \mathbf{R}^n$ be given

i) $(p, q, \Theta) \in D_{t+,x}^+ v(t_0, x_0)$ if and only if there exists $\varphi \in C^{1,2}([0, T] \times \mathbf{R}^n)$ satisfies

$$\left(\frac{\partial \varphi}{\partial t}(t_0, x_0), D_x \varphi(t_0, x_0), D^2 \varphi(t_0, x_0) \right) = (p(t_0, x_0), q(t_0, x_0), \Theta(t_0, x_0))$$

and such that $v - \varphi$ achieves its maximum at $(t_0, x_0) \in [0, T] \times \mathbf{R}^n$ from right side on t .

ii) $(p, q, \Theta) \in D_{t+,x}^- v(t_0, x_0)$ if and only if there exists $\varphi \in C^{1,2}([0, T] \times \mathbf{R}^n)$ satisfies

$$\left(\frac{\partial \varphi}{\partial t}(t_0, x_0), D_x \varphi(t_0, x_0), D^2 \varphi(t_0, x_0) \right) = (p(t_0, x_0), q(t_0, x_0), \Theta(t_0, x_0))$$

and such that $v - \varphi$ achieves its minimum at $(t_0, x_0) \in [0, T] \times \mathbf{R}^n$ from right side on t .

Moreover, if v has polynomial growth, i.e., if

$$|v(t, x)| \leq C(1 + |x|^k) \text{ for some } k \geq 1, \quad (t, x) \in [0, T] \times \mathbf{R}^n, \quad (2.2)$$

then φ can be chosen so that $\varphi, \varphi_t, D\varphi, D^2\varphi$ satisfy (2.2) (with possibly different constants C).

Under the assumptions [H1]-(H4), we have the following results.

Lemma 7. There exists a constant $C > 0$ such that, for all $0 \leq t \leq T$, $x, x' \in \mathbf{R}^d$,

$$\begin{cases} |V(t, x)| \leq C(1 + |x|), \\ |V(t, x) - V(t', x')| \leq C(|t - t'|^{\frac{1}{2}} + |x - x'|). \end{cases} \quad (2.3)$$

Moreover, V is a unique solution in the class of continuous functions which grow at most polynomially at infinity.

The proof can be seen in [2] or [4]. Then according to Definition 5 and Lemma 6, we have the following result.

Lemma 8. We claim that

$$\inf_{(p,q,\Theta,u) \in D_{t+,x}^+ v(t,x) \times U} [p + H(t, x, v, q, \Theta, u)] \geq 0, \quad \forall (t, x) \in [0, T] \times \mathbf{R}^d. \quad (2.4)$$

3 Stochastic Verification Theorem for Forward-Backward Controlled Systems

In this section, we give the stochastic verification theorem for Forward-Backward Controlled Systems within the framework of viscosity solutions. Firstly, we need the following two lemmas.

Lemma 9. Suppose that (H1)-(H4) hold. Let $(s, y) \in [0, T] \times \mathbf{R}^d$ be fixed and let $(X^{s,y;u}(\cdot), u(\cdot))$ be an admissible pair. Define processes

$$\begin{cases} z_1(r) \doteq b(r, X^{s,y;u}(r), u(r)), \\ z_2(r) \doteq \sigma(r, X^{s,y;u}(r), u(r)) \sigma^*(r, X^{s,y;u}(r), u(r)), \\ z_3(r) \doteq f(r, X^{s,y;u}(r), Y^{s,y;u}(r), Z^{s,y;u}(r), u(r)). \end{cases}$$

Then

$$\lim_{h \rightarrow 0+} \frac{1}{h} \int_t^{t+h} |z_i(r) - z_i(t)| dr = 0, \quad a.e. \ t \in [0, T], \ i = 1, 2, 3. \quad (3.1)$$

The proof can be found in [7] or [13].

Lemma 10. Let $g \in C([0, T])$. Extend g to $(-\infty, +\infty)$ with $g(t) = g(T)$ for $t > T$, and $g(t) = g(0)$, for $t < 0$. Suppose that there is a integrable function $\rho \in L^1(0, T; \mathbf{R})$ and some $h_0 > 0$, such that

$$\frac{g(t+h) - g(t)}{h} \leq \rho(t), \quad a.e. \ t \in [0, T], \quad h \leq h_0.$$

Then

$$g(\beta) - g(\alpha) \leq \int_{\alpha}^{\beta} \limsup_{h \rightarrow 0+} \frac{g(t+h) - g(t)}{h} dt, \quad \forall 0 \leq \alpha \leq \beta \leq T.$$

Proof. Applying Fatou's Lemma, we have

$$\begin{aligned} \int_{\alpha}^{\beta} \rho(r) dr &\geq \int_{\alpha}^{\beta} \limsup_{h \rightarrow 0+} \frac{g(r+h) - g(r)}{h} dr \\ &\geq \limsup_{h \rightarrow 0+} \int_{\alpha}^{\beta} \frac{g(r+h) - g(r)}{h} dr \\ &= \limsup_{h \rightarrow 0+} \frac{\int_{\alpha+h}^{\beta+h} g(r) dr - \int_{\alpha}^{\beta} g(r) dr}{h} \\ &= \limsup_{h \rightarrow 0+} \frac{\int_{\beta}^{\beta+h} g(r) dr - \int_{\alpha}^{\alpha+h} g(r) dr}{h} \\ &= g(\beta) - g(\alpha). \end{aligned}$$

□

The main result in this section is the following.

Theorem 11. (Verification Theorem) Assume that (H1)-(H4) hold. Let

$$v \in C([0, T] \times \mathbf{R}^d),$$

be a viscosity solution of the H-J-B equations (1.4), satisfying the following conditions:

$$\left\{ \begin{array}{l} i) v(t+h, x) - v(t, x) \leq C(1+|x|^m)h, \quad m \geq 0, \\ \quad \text{for all } x \in \mathbf{R}^d, 0 < t < t+h < T. \\ ii) v \text{ is semiconcave, uniformly in } t, \text{ i.e. there exists } C_0 \geq 0 \\ \quad \text{such that for every } t \in [0, T], v(t, \cdot) - C_0|\cdot|^2 \text{ is concave on } \mathbf{R}^d \end{array} \right. \quad (3.2)$$

Then we have

$$v(s, y) \leq J(s, y; u(\cdot)), \text{ for any } (s, y) \in (0, T] \times \mathbf{R}^d \text{ and any } u(\cdot) \in \mathcal{U}_{ad}(s, T). \quad (3.3)$$

Forthurmore, let $(s, y) \in (0, T] \times \mathbf{R}^d$ be fixed and let $(\bar{X}^{s,y;u}(\cdot), \bar{u}(\cdot))$ be an admissible pair for Problem C_{sy} such that there exist a function $\varphi \in C^{1,2}([0, T]; \mathbf{R}^d)$ and a triple

$$(\bar{p}, \bar{q}, \bar{\Theta}) \in (L^2_{\mathcal{F}_t}(s, T; \mathbf{R}) \times L^2_{\mathcal{F}_t}(s, T; \mathbf{R}^d) \times L^2_{\mathcal{F}_t}(s, T; \mathbf{S}^d)) \quad (3.4)$$

satisfying

$$\left\{ \begin{array}{l} (\bar{p}(t), \bar{q}(t), \bar{\Theta}(t)) \in D_{t+,x}^+ v(t, \bar{X}^{s,y;u}(t)), \\ (\frac{\partial \varphi}{\partial t}(t, \bar{X}^{s,y;u}(t)), D_x \varphi(t, \bar{X}^{s,y;u}(t)), D^2 \varphi(t, \bar{X}^{s,y;u}(t))) = (\bar{p}(t), \bar{q}(t), \bar{\Theta}(t)), \\ \varphi(t, x) \geq v(t, x) \quad \forall (t_0, x_0) \neq (t, x), \text{ a.e. } t \in [0, T], \text{ } P\text{-a.s.} \end{array} \right. \quad (3.5)$$

and

$$\mathbf{E} \left[\int_s^T [\bar{p}(t) + H(t, \bar{X}^{s,y;u}(t), \bar{\varphi}(t), \bar{p}(t), \bar{\Theta}(t), \bar{u}(t))] dt \right] \leq 0, \quad (3.6)$$

where

$$\bar{\varphi}(t) = \varphi(t, \bar{X}^{s,y;u}(t)).$$

Then $(\bar{X}^{s,y;u}(\cdot), \bar{u}(\cdot))$ is an optimal pair for the problem C_{sy} .

Proof. Firstly, (3.3) follows from the uniqueness of viscosity solutions of the H-J-B equations (1.4). It remains to show that $(\bar{X}^{s,y;u}(\cdot), \bar{u}(\cdot))$ is an optimal.

We now fix $t_0 \in [s, T]$ such that (3.4) and (3.5) hold at t_0 and (3.1) holds at t_0 for

$$\left\{ \begin{array}{l} z_1(\cdot) = \bar{b}(\cdot), \\ z_2(\cdot) = \bar{\sigma}(\cdot) \bar{\sigma}(\cdot)^* \\ z_3(\cdot) = \bar{f}(\cdot). \end{array} \right.$$

We claim that the set of such points is of full measure in $[s, T]$ by Lemma 9. Now we fix $\omega_0 \in \Omega$ such that the regular conditional probability $\mathbf{P}(\cdot | \mathcal{F}_{t_0}^s)(\omega_0)$, given $\mathcal{F}_{t_0}^s$ is well defined. In this new probability space, the random variables

$$\bar{X}^{s,y;u}(t_0), \bar{p}(t_0), \bar{q}(t_0), \bar{\Theta}(t_0)$$

are almost surely deterministic constants and equal to

$$\bar{X}^{s,y;u}(t_0, \omega_0), \bar{p}(t_0, \omega_0), \bar{q}(t_0, \omega_0), \bar{\Theta}(t_0, \omega_0),$$

respectively. We remark that in this probability space the Brownian motion W is still the a standard Brownian motion although now $W(t_0) = W(t_0, \omega_0)$ almost surely. The space is now equipped with a new filtration $\{\mathcal{F}_r^s\}_{s \leq r \leq T}$ and the control process $\bar{u}(\cdot)$ is adapted to this new filtration. For P -a.s. ω_0 the process $\bar{X}^{s,y;u}(\cdot)$ is a solution of (1.1) on $[t_0, T]$ in $(\Omega, \mathcal{F}, \mathbf{P}(\cdot | \mathcal{F}_{t_0}^s)(\omega_0))$ with the inial condition $\bar{X}^{s,y;u}(t_0) = \bar{X}^{s,y;u}(t_0, \omega_0)$.

Then on the probability space $(\Omega, \mathcal{F}, \mathbf{P}(\cdot | \mathcal{F}_{t_0}^s)(\omega_0))$, we are going to apply Itô's formula to φ on $[t_0, t_0 + h]$ for any $h > 0$,

$$\begin{aligned} & \varphi(t_0 + h, \bar{X}^{s,y;u}(t_0 + h)) - \varphi(t_0, \bar{X}^{s,y;u}(t_0)) \\ &= \int_{t_0}^{t_0+h} \left[\frac{\partial \varphi}{\partial t}(r, \bar{X}^{s,y;u}(r)) + \langle D_x \varphi(r, \bar{X}^{s,y;u}(r)), \bar{b}(r) \rangle \right. \\ & \quad \left. + \frac{1}{2} \text{tr} \{ \bar{\sigma}(r)^* D_{xx} \varphi(r, \bar{X}^{s,y;u}(r)) \bar{\sigma}(r) \} \right] dr \\ & \quad + \int_{t_0}^{t_0+h} \langle D_x \varphi(r, \bar{X}^{s,y;u}(r)), \bar{\sigma}(r) \rangle dW_r. \end{aligned}$$

Taking conditional expectation value $\mathbf{E}^{\mathcal{F}_{t_0}^s}(\cdot)(\omega_0)$, dividing both sides by h , and using (3.5), we have

$$\begin{aligned} & \frac{1}{h} \mathbf{E}^{\mathcal{F}_{t_0}^s}(\omega_0) [v(t_0 + h, \bar{X}^{s,y;u}(t_0 + h)) - v(t_0, \bar{X}^{s,y;u}(t_0))] \\ & \leq \frac{1}{h} \mathbf{E}^{\mathcal{F}_{t_0}^s}(\omega_0) [\varphi(t_0 + h, \bar{X}^{s,y;u}(t_0 + h)) - \varphi(t_0, \bar{X}^{s,y;u}(t_0))] \\ & = \frac{1}{h} \mathbf{E}^{\mathcal{F}_{t_0}^s}(\omega_0) \left\{ \int_{t_0}^{t_0+h} \left[\frac{\partial \varphi}{\partial t}(r, \bar{X}^{s,y;u}(r)) + \langle D_x \varphi(r, \bar{X}^{s,y;u}(r)), \bar{b}(r) \rangle \right. \right. \\ & \quad \left. \left. + \frac{1}{2} \text{tr} \{ \bar{\sigma}(r)^* D_{xx} \varphi(r, \bar{X}^{s,y;u}(r)) \bar{\sigma}(r) \} \right] dr \right\} \end{aligned} \tag{3.7}$$

Letting $h \rightarrow 0$, and employing the similar delicate method as in the proof of Theorem 4.1 of Gozzi et al. [12], we have

$$\begin{aligned} & \frac{1}{h} \limsup_{h \rightarrow 0+} \mathbf{E}^{\mathcal{F}_{t_0}^s}(\omega_0) [v(t_0 + h, \bar{X}^{s,y;u}(t_0 + h)) - v(t_0, \bar{X}^{s,y;u}(t_0))] \\ & \leq \frac{\partial \varphi}{\partial t}(t_0, \bar{X}^{s,y;u}(t_0, \omega_0)) + \langle D_x \varphi(t_0, \bar{X}^{s,y;u}(t_0, \omega_0)), \bar{b}(t_0) \rangle \\ & \quad + \frac{1}{2} \text{tr} \{ \bar{\sigma}(t_0)^* D_{xx} \varphi(t_0, \bar{X}^{s,y;u}(t_0, \omega_0)) \bar{\sigma}(t_0) \} \\ & = \bar{p}(t_0, \omega_0) + \langle \bar{q}(t_0, \omega_0), \bar{b}(t_0) \rangle + \frac{1}{2} \text{tr} \{ \bar{\sigma}(t_0)^* \bar{\Theta}(t_0, \omega_0) \bar{\sigma}(t_0) \} \end{aligned}$$

By (3.2), we know, from [12], that there exist

$$\rho \in L^1(t_0, T; \mathbf{R}) \text{ and } \rho_1 \in L^1(\Omega; \mathbf{R})$$

such that

$$\mathbf{E} \left[\frac{1}{h} [v(t+h, \bar{X}^{s,y;u}(t+h)) - v(t, \bar{X}^{s,y;u}(t))] \right] \leq \rho(t), \text{ for } h \leq h_0, \text{ for some } h_0 > 0, \quad (3.8)$$

and

$$\begin{aligned} & \mathbf{E}^{\mathcal{F}_{t_0}^s(\omega_0)} \left[\frac{1}{h} [v(t+h, \bar{X}^{s,y;u}(t+h)) - v(t, \bar{X}^{s,y;u}(t))] \right] \\ & \leq \rho_1(\omega_0), \text{ for } h \leq h_0, \text{ for some } h_0 > 0. \end{aligned} \quad (3.9)$$

holds, respectively. By virtue of Fatou's Lemma, noting (3.9), we obtain

$$\begin{aligned} & \limsup_{h \rightarrow 0+} \frac{1}{h} \mathbf{E} [v(t_0+h, \bar{X}^{s,y;u}(t_0+h)) - v(t_0, \bar{X}^{s,y;u}(t_0))] \\ & = \limsup_{h \rightarrow 0+} \frac{1}{h} \mathbf{E} \left[\mathbf{E}^{\mathcal{F}_{t_0}^s(\omega_0)} \{v(t_0+h, \bar{X}^{s,y;u}(t_0+h)) - v(t_0, \bar{X}^{s,y;u}(t_0))\} \right] \\ & \leq \mathbf{E} \left[\limsup_{h \rightarrow 0+} \frac{1}{h} \mathbf{E}^{\mathcal{F}_{t_0}^s(\omega_0)} \{v(t_0+h, \bar{X}^{s,y;u}(t_0+h)) - v(t_0, \bar{X}^{s,y;u}(t_0))\} \right] \\ & \leq \mathbf{E} \left[\bar{p}(t_0) + \langle \bar{q}(t_0), \bar{b}(t_0) \rangle + \frac{1}{2} \text{tr} \{ \bar{\sigma}(t_0)^* \bar{\Theta}(t_0) \bar{\sigma}(t_0) \} \right], \end{aligned} \quad (3.10)$$

for a.e. $t_0 \in [s, T]$. Then the rest of the proof goes exactly as in [11]. We apply Lemma 10 to

$$g(t) = \mathbf{E} [v(t, \bar{X}^{s,y;u}(t))],$$

using (3.8), (3.6) and (3.10) to get

$$\begin{aligned} & \mathbf{E} [v(T, \bar{X}^{s,y;u}(T)) - v(s, y)] \\ & \leq \mathbf{E} \left\{ \int_s^T \left[\bar{p}(t) + \langle \bar{q}(t), \bar{b}(t) \rangle + \frac{1}{2} \text{tr} [\bar{\sigma}(t)^* \bar{\Theta}(t) \bar{\sigma}(t)] \right] dt \right\} \\ & \leq -\mathbf{E} \left[\int_s^T \bar{f}(t) dt \right]. \end{aligned}$$

From this we claim that

$$\begin{aligned} v(s, y) & \geq \mathbf{E} \left[v(T, \bar{X}^{s,y;u}(T)) + \int_s^T \bar{f}(t) dt \right] \\ & = \mathbf{E} \left[\Phi(\bar{X}^{s,y;u}(T)) + \int_s^T \bar{f}(t) dt \right]. \end{aligned}$$

Thus, combining the above with the first assertion (3.3), we prove the $(\bar{X}^{s,y;u}(\cdot), \bar{u}(\cdot))$ is an optimal pair. The proof is complete. \square

Remark 12. The condition (3.6) is just equivalent to the following:

$$\begin{aligned}\bar{p}(t) &= \min_{u \in U} H(t, \bar{X}^{s,y;u}(t), \bar{\varphi}(t), \bar{q}(t), \bar{\Theta}(t), u) \\ &= H(t, \bar{X}^{s,y;u}(t), \bar{\varphi}(t), \bar{q}(t), \bar{\Theta}(t), \bar{u}(t)), \\ \text{a.e. } t &\in [s, T], \text{ } P\text{-a.s.},\end{aligned}\tag{3.11}$$

where $\bar{\varphi}(t)$ is defined in Theorem 11. This is easily seen by recalling the fact that v is the viscosity solution of (1.4):

$$\bar{p}(t) + \min_{u \in U} H(t, \bar{X}^{s,y;u}(t), \bar{\varphi}(t), \bar{q}(t), \bar{\Theta}(t), u) \geq 0,$$

which yields (3.11) under (3.6).

4 Optimal Feedback Controls

In this section, we describe the method to construct optimal feedback controls by the verification Theorem 11 obtained. First, let us recall the definition of admissible feedback controls.

Definition 13. A measurable function \mathbf{u} from $[0, T] \times \mathbf{R}^d$ to U is called an admissible feedback control if for any $(s, y) \in [0, T] \times \mathbf{R}^d$ there is a weak solution $X^{s,y;u}(\cdot)$ of the following SDEs:

$$\begin{cases} dX^{s,y;u}(t) = b(t, X^{s,y;u}(t), \mathbf{u}(t)) dt + \sigma(t, X^{s,y;u}(t), \mathbf{u}(t)) dW(t), \\ dY^{s,y;u}(t) = -f(t, X^{s,y;u}(t), Y^{s,y;u}(t), \mathbf{u}(t)) dt + dM^{s,y;u}(t), \\ X^{s,y;u}(s) = x, \quad Y^{s,y;u}(T) = \Phi(X^{s,y;u}(T)), \end{cases}\tag{4.1}$$

where $M^{s,y;u}$ is an \mathbf{R} -valued $\mathbb{F}^{s,y;u}$ -adapted right continuous and left limit martingale vanishing in $t = 0$ which is orthogonal to the driving Brownian motion W . Here $\mathbb{F}^{s,y;u} = (\mathcal{F}_t^{X^{s,y;u}})_{t \in [s, T]}$ is the smallest filtration and generated by $X^{s,y;u}$, which is such that $X^{s,y;u}$ is $\mathbb{F}^{s,y;u}$ -adapted. Obviously, $M^{s,y;u}$ is a part of the solution of BSDEs of (4.1). Simultaneously, we suppose that f satisfies the Lipschitz condition.

$$\begin{aligned}\left| f(t, x, y, u) - f(t, x', y', u') \right| &\leq L \left(\left| x - x' \right| + \left| y - y' \right| + \left| u - u' \right| \right) \\ x, x' &\in \mathbf{R}^d, y, y' \in \mathbf{R}, u, u' \in U.\end{aligned}$$

An admissible feedback control \mathbf{u}^* is called optimal if $(X^*(\cdot; s, y), \mathbf{u}^*(\cdot, X^*(\cdot; s, y)))$ is optimal for the problem $C_{s,y}$ for each (s, y) is a solution of (4.1) corresponding to \mathbf{u}^* .

Theorem 14. Let \mathbf{u}^* be an admissible feedback control and p^*, q^* , and Θ^* be measurable functions satisfying

$$(p^*(t, x), q^*(t, x), \Theta(t, x)) \in D_{t+,x}^+ V(t, x)\tag{4.2}$$

for all $(t, x) \in [0, T] \times \mathbf{R}^d$. If

$$\begin{aligned}
& p^*(t, x) + H(t, x, V(t, x), q^*(t, x), \Theta^*(t, x), \mathbf{u}^*(t, x)) \\
&= \inf_{(p, q, \Theta, u) \in D_{t+, x}^+ V(t, x) \times U} [p + H(t, x, V(t, x), q, \Theta, u)] \\
&= 0
\end{aligned} \tag{4.3}$$

for all $(t, x) \in [0, T] \times \mathbf{R}^d$, then \mathbf{u}^* is optimal.

Proof From Theorem 11, we get the desired result. \square

Remark 15. *Actually, it is fairly easy to check that in Eq.(4.1), $Y^{s, y; u}(\cdot)$ is determined by $(X^{s, y; u}(\cdot), u(\cdot))$. Hence, we need to investigate the conditions imposed in Theorem 11 to ensure the existence and uniqueness of $X^{s, y; u}(\cdot)$ in law and the measurability of the multifunctions $(t, x) \rightarrow D_{t+, x}^+ V(t, x)$ to obtain $(p^*(t, x), q^*(t, x), \Theta(t, x)) \in D_{t+, x}^+ V(t, x)$ that minimizes (4.3) by virtue of the celebrated Filippov's Lemma. The rest parts we can get from [7] or [13].*

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References

- [1] E. Pardoux, S. Peng, Adapted solution of a backward stochastic differential equation, System Control Lett. 14 (1) (1990) 55-61.
- [2] Z. Wu, Z. Yu, Dynamic programming principle for one kind of stochastic recursive optimal control problem and Hamilton-Jacobi-Bellman equation, SIAM J. Control Optim. 47 (5) (2008) 2616-2641.
- [3] J. Li, S. Peng, Stochastic optimization theory of backward stochastic differential equations with jumps and viscosity solutions of Hamilton-Jacobi-Bellman equations, Nonlinear Analysis 70 (2009) 1776-1796.
- [4] R. Buckdahn, J. Li, Stochastic differential games and viscosity solutions of Hamilton-Jacobi-Bellman-Isaacs equations, SIAM J. Control Optim. 47 (1) (2008) 444-475.
- [5] R. Buckdahn, B. Labed, C. Rainer, L. Tamer, Existence of an optimal control for stochastic control system with nonlinear cost functional, Stoch. Stoch. Rep. 82 (3) (2010) 241-256.
- [6] X. Zhou, Verification theorem within the framework of viscosity solutions, J. Math. Anal. Appl. 176 (1993) 208-225.

- [7] X. Zhou, J. Yong, X. Li, Stochastic verification theorems within the framework of viscosity solutions, *SIAM J. Control Optim.* 35 (1) (1997) 243-253.
- [8] S. Federico, B. Goldys, and F. Gozzi, HJB equations for the optimal control of differential equations with delays and state constraints, II: Optimal feedbacks and approximations, preprint, 2009; available online from <http://arxiv.org/abs/0907.1603>.
- [9] M. Kohlmann, P. Renner, Optimal control of diffusions: A verification theorem for viscosity solutions, *System Control Lett.* 28 (1996) 247-253.
- [10] M. G. Crandall, H. Ishii, P. L. Lions, User's guide to viscosity solutions of second order partial differential equations, *Bull. Amer. Math. Soc.*, 27 (1992) 1-67.
- [11] F. Gozzi, A. Swiech, X. Zhou, A corrected proof of the stochastic verification theorem within the framework of viscosity solutions, *SIAM J. Control Optim.* 43 (6) (2005) 2009-2019.
- [12] F. Gozzi, A. Swiech, X. Zhou, Erratum: "A corrected proof of the stochastic verification theorem within the framework of viscosity solutions", *SIAM J. Control Optim.* 48 (6) (2010) 4177-4179.
- [13] J. Yong, X. Zhou, *Stochastic controls: Hamiltonian systems and HJB equations*. New York: Springer-Verlag, (1999).